B.Sc. (Math) part III

paper – VI

Topic : Solvable Groups

Solvable group

Def :– A Group is said to be solvable if we can find a finite chain of subgroup

$G = N_0 \supseteq N_1 \supseteq N_2 \supseteq \cdots \supseteq N_K = (e)$

such that each N_i is a normal subgroup of N_{i-1} and each quotient group $\frac{N_{i-1}}{N_i}$ is abelian. The above mentioned series then is referred to as solvable series of G.

Normal series of a group

Def :– A finite sequence of subgroup

$G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots \supseteq G_K = (e)$

of a group G is called a sub normal series of G if G_{i+1} is normal subgroup of G_i $\forall i = 0, 1, 2 \cdots K$. The quotients groups G_i / G_{i+1} are called the factor groups of the subnormal series. Further if each G_i is normal subgroup of G itself then the series is said to be a normal of G.

Theorem ① prove that every abelian group is solvable.

proof:- Let G is an abelian group ~~Clearly $N_1=(e)$~~ we can take

$$N_0 = G \text{ and } N_1=(e)$$

Then $G = N_0 \supseteq N_1 = (e)$ is a solvable series of G. Clearly $N_1=(e)$ is a normal subgroup of $N_0=G$ because if a is any element of G then

$$a^{-1}ea = a^{-1}a = e \in (e)$$

~~Again~~ since G is abelian the quotient group $\frac{N_0}{N_1} = G/(e)$ is also abelian. we know that every quotient group of an abelian group is abelian

So G is solvable group.

Def:- Commutator subgroup of a group.

Let G be a group and $a,b \in G$. The element $aba^{-1}b^{-1}$ is called the commutator of the ordered pair (a,b)

Let $U = \{aba^{-1}b^{-1} : a,b \in G\}$

Theorem ② prove that a subgroup of a solvable group solvable.

proof:- Let H be any subgroup of a Solvable group G, let

$$G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots \supseteq G_n = (e)$$

be a Solvable series of G, we claim that

$$H = H_0 \supseteq (H \cap G_1) \supseteq (H \cap G_2) \supseteq \cdots \supseteq (H \cap G_n) = (e) \text{ - (1)}$$

is a solvable series for H since for $i = 0, 1, 2 \cdots n-1$, G_{i+1} is normal G_i therefore

$H_{i+1} = H \cap G_{i+1}$ is normal in $H_i = H \cap G_i$

we define a mapping

$f: H_i \to G_i / G_{i+1}$ such that

$f(x) = xG_{i+1} \ \forall \ x \in H_i$

Since $H_i \subseteq G_i$ therefore $x \in H_i \Rightarrow x \in G_i$

Consequently the coset xG_{i+1} is an element of quotient group G_i/G_{i+1} and thus mapping f is well defined

Also, if $x, y \in H_i$ then

$f(xy) = (xy)G_{i+1}$

$= (xG_{i+1})(yG_{i+1})$ [$\because x, y \in G_i$ and G_{i+1} is normal in G_i]

$= f(x)f(y)$

Therefore the mapping f is a homomorphism of H_i into G_i/G_{i+1}

$x \in$ Kernal $f \Leftrightarrow f(x) = G_{i+1}$. [
The identities of G_i / G_{i+1} is G_{i+1}

$\Leftrightarrow G_{i+1} = G_{i+1} \Leftrightarrow x \in G_{i+1}$

$\Leftrightarrow x \in H \cap G_{i+1}$ since $x \in H_i \subseteq H$

Therfor $\ker f = H \cap G_{i+1} = H_{i+1}$

Hence by fundamental theorem on homomorphism of group have

$$H_i / H_{i+1} \cong f(H_i)$$

But $f(H_i)$ is a subgroup of G_i / G_{i+1} which is abelian.

Therefor $f(H_i)$ is also abelian

Consequently H_i / H_{i+1} is also abelian because it is isomorphic to $f(H_i)$. Hence (1) is a solvable series for H and thus H is a solvable group

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